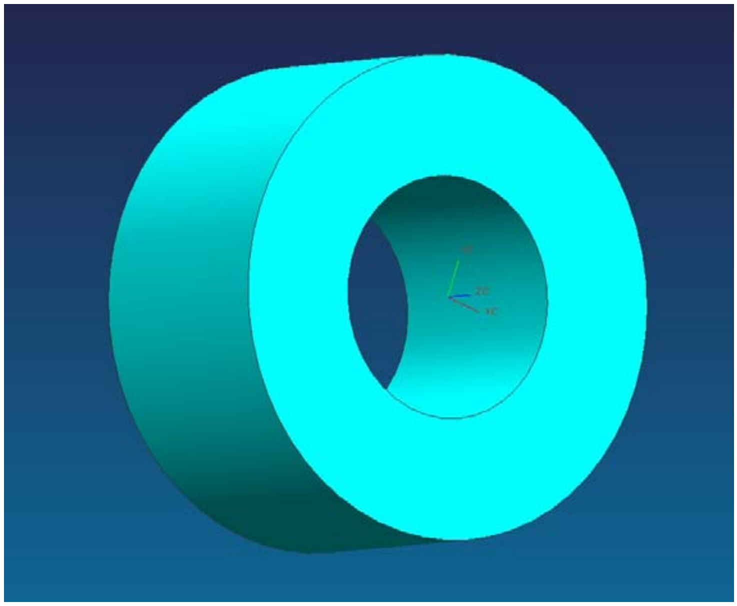
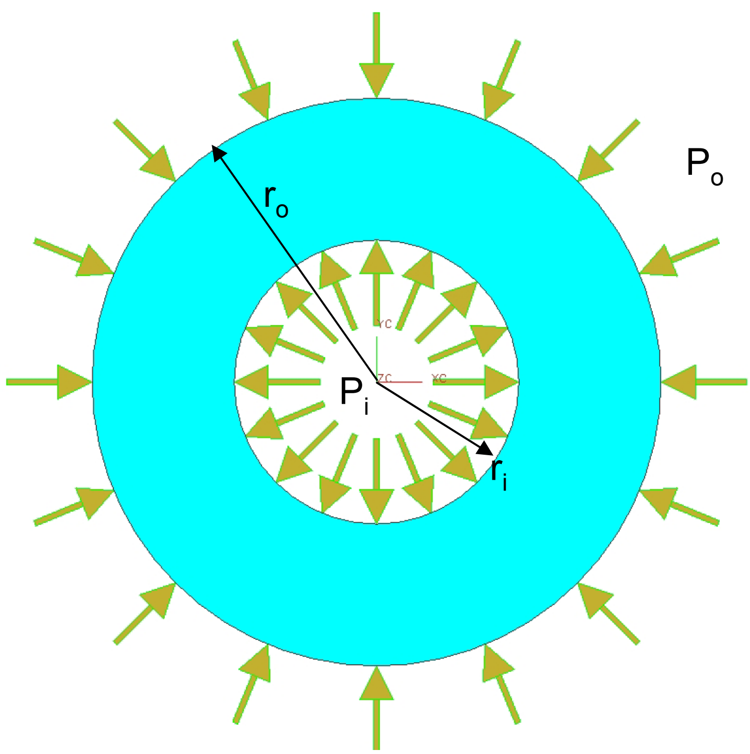
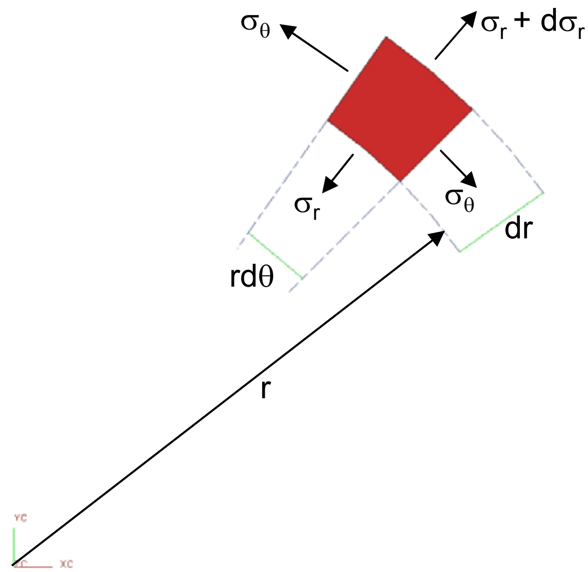
Thick Walled Cylinders



Consider a thick walled cylinder with open ends as shown above. It is loaded by internal pressure P_i and external pressure P_o as seen below. It has inner radius r_i and outer radius r_o.





Now consider and element at radius r and defined by an angle increment $d\theta$ and a radial increment dr. By circular symmetry, the stresses σ_θ and σ_r are functions of r only, not θ and the shear stress on the element must be zero. For an element of unit thickness, radial force equilibrium gives:

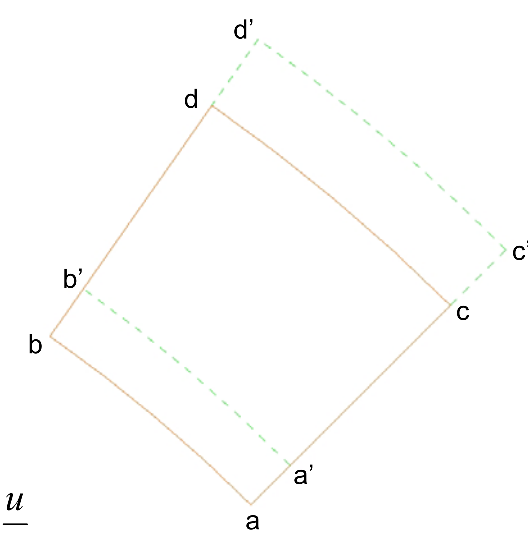
$$(\sigma_r + d\sigma_r)(r + dr)d\theta = \sigma_r r d\theta + \sigma_\theta d\theta\, dr$$

Ignoring second order terms gives:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r + \sigma_\theta}{r} = 0 \quad \cdots\cdots\cdots (1)$$

Assuming that there are no body forces.

Now consider strains in the element. By symmetry there is no θ displacement v. there is only a radial displacement u given by line aa'. Point c is displaced radially by (u + du) given by line cc'. As the original radial length of the element is dr (line ac), the radial strain is:



$$\varepsilon_r = \frac{u + du - u}{dr} = \frac{du}{dr}$$

Line ab has length $rd\theta$ and line a'b' has length $(r + u)d\theta$. The tangential strain is thus:

$$\varepsilon_\theta = \frac{(r + u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

As the ends are open, $\sigma_z = \sigma_3 = 0$ and we thus have plane stress conditions.
From Hooke's law we get:

$$\varepsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \sigma_r)$$

Solving for the stresses gives:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \quad and \quad \sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substituting into equation above yields:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Which has solution : $u = C_1 r + \frac{C_2}{r}$

Giving the stresses as:

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right] \cdots\cdots\cdots (2)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right] \cdots\cdots\cdots (3)$$

The boundary conditions are: $\sigma_r(r_i) = -P_i \quad and \quad \sigma_r(r_o) = -P_o$

This yields the integration constants:

$$C_1 = \frac{1-\nu}{E}\left(\frac{r_i^2 P_i - r_o^2 P_o}{r_o^2 - r_i^2}\right) \quad and \quad C_2 = \frac{1-\nu}{E}\left(\frac{r_i^2 r_o^2 (P_i - P_o)}{r_o^2 - r2_i}\right)$$

Giving the stresses as a function of radius:

$$\sigma_r = \frac{r_i^2 P_i - r_o^2 P_o}{(r_o^2 - r_i^2)} - \frac{(P_i - P_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2} \qquad \sigma_\theta = \frac{r_i^2 P_i - r_o^2 P_o}{(r_o^2 - r_i^2)} + \frac{(P_i - P_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2}$$

These are known as Lamé's equations.

From equations 2 and 3 above we can see that the sum of the radial and tangential stresses is constant, regardless of radius: $\sigma_r + \sigma_\theta = 2EC_1/(1-\nu)$

Hence the longitudinal strain is also constant since:

$\varepsilon_z = -\dfrac{\nu}{E}(\sigma_r + \sigma_\theta) =$ constant. Hence we get $\sigma_z = E\varepsilon_z$ = constant = c

If the ends of the cylinder and open and free we have $F_z = 0$, hence:

$$\int_{r_i}^{r_o} \sigma_z \cdot 2\pi r\, dr = \pi c\left(r_p^2 - r_i^2\right) = 0$$

or c = σ_z = 0 as we assumed.

If the cylinder has closed ends, the axial stress can be found separately using only force equilibrium considerations as was done for the thin walled cylinder. The result is then simply superimposed on the above equations.

The pressure P_i acts on area given by $\pi\, r_i^2$.

The pressure P_o acts on area given by πr_o^2.

The axial stress σ_z acts on an area given by $\pi(r_o^2 - r_i^2)$

Force equilibrium then gives:

$$\sigma_z = \left(\frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}\right)$$

The following is a summary of the equations used to determine the stresses found in thick walled cylindrical pressure vessels. In the most general case the vessel is subject to both internal and external pressures. Most vessels also have closed ends - this results in an axial stress component.

Principal stresses at radius r :

$$\sigma_1 = \sigma_\theta = -K + C/r^2 \quad : \quad \sigma_2 = \sigma_r = -K + C/r^2$$

And, if the ends are closed, $\sigma_3 = \sigma_{axial} = -K$

Where: $$C = (P_o - P_i)\left[\frac{r_i^2 r_o^2}{r_0^2 - r_i^2}\right] \quad : \quad K = \frac{(P_o r_0^2 - P_i r_i^2)}{r_0^2 - r_i^2}$$

(a) Internal Pressure only (P_o = 0):

$$\sigma_\theta = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left[1 - \frac{r_o^2}{r^2}\right] \quad : \quad \sigma_r = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left[1 + \frac{r_o^2}{r^2}\right] \quad : \quad \sigma_z = \frac{P_i r_i^2}{r_o^2 - r_i^2}$$

At inside surface, r = r_i:

$$\sigma_\theta = P_i\left[\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}\right] \quad : \quad \sigma_r = -P_i \quad : \quad \sigma_z = \frac{P_i r_i^2}{r_o^2 - r_i^2}$$

At outside surface, r = $r_{o:}$

$$\sigma_\theta = \left[\frac{2 P_i r_i^2}{r_o^2 - r_i^2}\right] \quad : \quad \sigma_r = 0 \quad : \quad \sigma_z = \frac{P_i r_i^2}{r_o^2 - r_i^2}$$

(b) External Pressure only (P_i = 0):

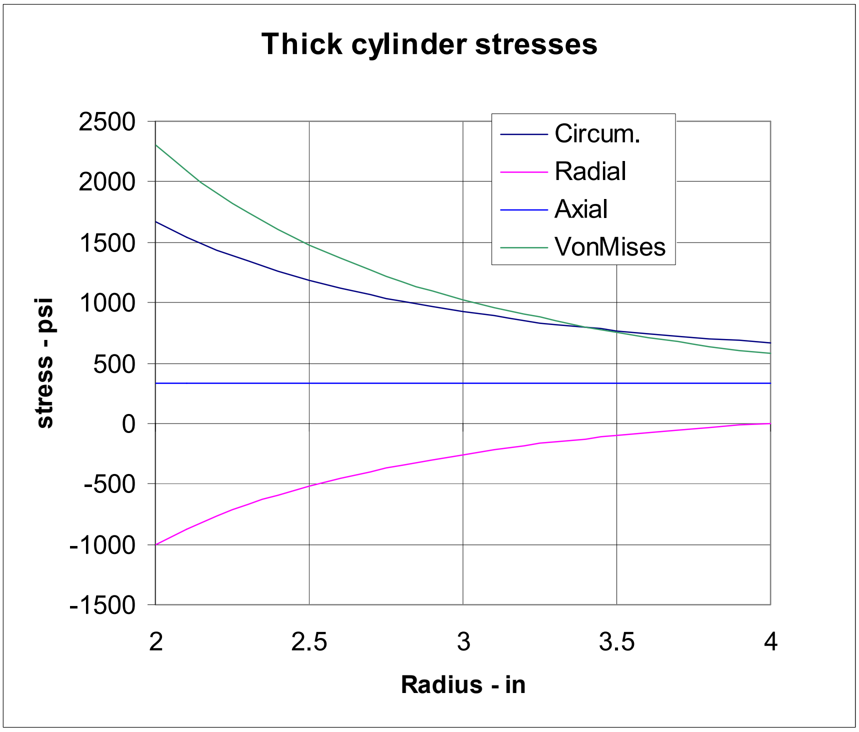
$$\sigma_\theta = \frac{-P_o r_o^2}{r_o^2 - r_i^2}\left[1 + \frac{r_i^2}{r^2}\right] \quad : \quad \sigma_r = \frac{-P_o r_o^2}{r_o^2 - r_i^2}\left[1 - \frac{r_i^2}{r^2}\right] \quad : \quad \sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$

At inside surface, r = r_i:

$$\sigma_\theta = \left[\frac{-2 P_o r_o^2}{r_o^2 - r_i^2}\right] \quad : \quad \sigma_r = 0 \quad : \quad \sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$

At outside surface, r = r_o:

$$\sigma_\theta = -P_o\left[\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}\right] \quad : \quad \sigma_r = -P_o \quad : \quad \sigma_z = \frac{-P_o r_o^2}{r_o^2 - r_i^2}$$



Example of cylinder with P_i = 1000 psi, r_i = 2" and r_o=4"

Note that in all cases the greatest magnitude of direct stress is the tangential stress at the in-side surface. The maximum magnitude of shear stress also occurs at the inside surface.

(c) Press and shrink fits

When a press or shrink fit is used between 2 cylinders of the same material, an interface pressure p_i is developed at the junction of the cylinders. If this pressure is calculated, the stresses in the cylinders can be found using the above equations. The pressure is:

$$p_i = \frac{E\delta}{b}\left[\frac{\left(c^2 - b^2\right)\left(b^2 - a^2\right)}{2b^2\left(c^2 - a^2\right)}\right]$$

Where:

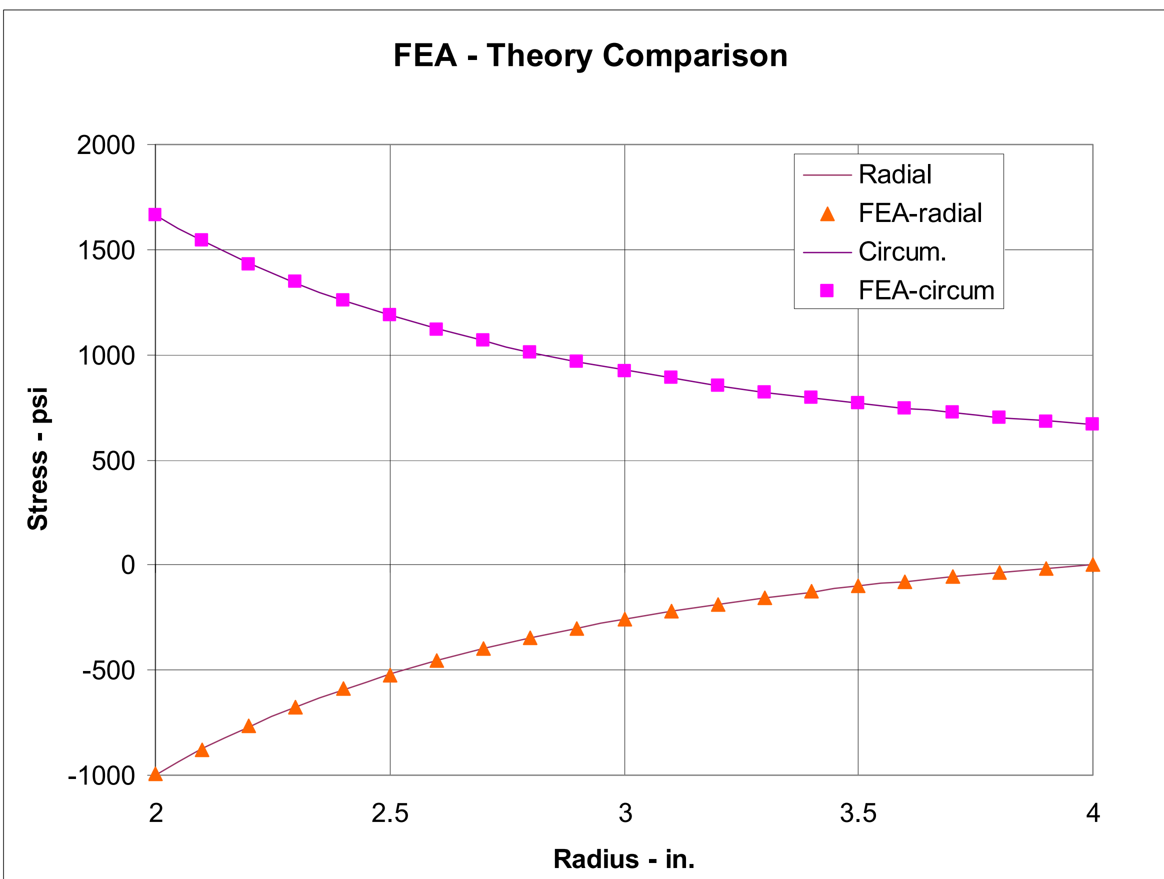
E = Young's Modulus
δ = radial interference between the two cylinders
a = inner radius of the inner cylinder
b = outer radius of inner cylinder and inner radius of outer cylinder
c = outer radius of outer cylinder

It is assumed that δ is very small compared to the radius b and that there are no axial stresses. Thus we have $\delta = b_{inner} - b_{outer}$. Note that this small difference in the radii is ignored in the above equation.

Comparison of thin and thick wall cylinder equations

P_i	r_i	r_o	t	Thin wall theory σ_c	σ_r	σ_a	Thick wall theory σ_c	σ_r	σ_a	%age difference in σ_c	%age difference in σ_a	r_o/t
1	0.990	1	0.01	100	0	50	99.503	-1	49.251	0.50%	1.52%	100
1	0.975	1	0.025	40	0	20	39.506	-1	19.253	1.25%	3.88%	40
1	0.950	1	0.05	20	0	10	19.513	-1	9.2564	2.50%	8.03%	20
1	0.900	1	0.1	10	0	5	9.526	-1	4.2632	4.97%	17.28%	10
1	0.875	1	0.125	8	0	4	7.533	-1	3.2667	6.19%	22.45%	8
1	0.850	1	0.15	6.6667	0	3.3333	6.207	-1	2.6036	7.40%	28.03%	6.6667
1	0.800	1	0.2	5	0	2.5	4.556	-1	1.7778	9.76%	40.62%	5
1	0.750	1	0.25	4	0	2	3.571	-1	1.2857	12.00%	55.56%	4
1	0.700	1	0.3	3.3333	0	1.6667	2.922	-1	0.9608	14.09%	73.47%	3.3333
1	0.650	1	0.35	2.8571	0	1.4286	2.463	-1	0.7316	15.99%	95.27%	2.8571
1	0.600	1	0.4	2.5	0	1.25	2.125	-1	0.5625	17.65%	122.22%	2.5
1	0.550	1	0.45	2.2222	0	1.1111	1.867	-1	0.4337	19.00%	156.20%	2.2222
1	0.500	1	0.5	2	0	1	1.667	-1	0.3333	20.00%	200.00%	2

All stresses are calculated at the inner radius and are for a cylinder with closed ends and internal pressure only.



Problems on Thick Cylinders.

1. A steel cylinder is 160 mm ID and 320 mm OD. If it is subject to an internal pressure of 150 MPa, determine the radial and tangential stress distributions and show the results on a plot (using a spreadsheet). Determine the maximum shear stress in the cylinder. Assume it has closed ends.
(σ_t = 250 to 100 MPa, σ_r = 0 to ±150 MPa, τ_{max} = 200 MPa.)

2. A cylinder is 150 mm ID and 450 mm OD. The internal pressure is 160 MPa and the external pressure is 80 MPa. Find the maximum radial and tangential stresses and the maximum shear stress. The ends are closed.
(σ_t = 20 to ±60 MPa, σ_r = ±80 to ±160 MPa, τ_{max} = 90 MPa.)

3. A cylinder has an ID of 100 mm and an internal pressure of 50 MPa. Find the needed wall thickness if the factor of safety n is 2.0 and the yield stress is 250 MPa. Use the maximum shear stress theory, i.e. maximum shear stress = yield strength/2n.
(wall = 61.8 mm thick)

4. A 400 mm OD steel cylinder with a nominal ID of 240 mm is shrunk onto another steel cylinder of 240 mm OD and 140 mm ID. The radial interference δ is 0.3 mm. Use Young's Modulus E = 200 GPa and Poisson's Ratio n = 0.3. Find the interface pressure p_i and plot the radial and tangential stresses in both cylinders. Then find the maximum internal pressure which may be applied to the assembly if the maximum tangential stress in the inside cylinder is to be no more than 140 MPa.
(p_i = ±120.3 MPa. :: inner cylinder : σ_t = ±365 to ±244 MPa, σ_r = 0 to ±120.3 MPa. :: outer cylinder : σ_t = 256 to 135 MPa, σ_r = ±120.3 to 0 MPa. :: maximum internal pressure = 395 MPa.)

5. A cylinder with closed ends has outer diameter D and a wall thickness t = 0.1D. Determine the %age error involved in using thin wall cylinder theory to calculate the maximum value of tangential stress and the maximum shear stress in the cylinder.
(tangential stress ± 9.75% : max. shear stress ± 11.1%)